Cardinal characteristics and strong compactness

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Strong compactness vs Supercompactness

Definition

- κ is strongly compact iff for all $\lambda \geqslant \kappa$ there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and there is $s \in M$ with $|s|^M < j(\kappa)$ and $j``\lambda \subseteq s$.
- κ is supercompact iff for all $\lambda \geqslant \kappa$ there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and ${}^{\lambda}M \subseteq M$.

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Is it consistent/possible to control the cardinal characteristics of a non-supercompact strongly compact cardinal?

- The result in this talk is a stepping stone to a positive answer.

Cardinal characteristics

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- There is ongoing research on generalising the known cardinal characteristics of the continuum.
- It is connected to the study of the generalised Baire space κ^κ or 2^κ.
- Sometimes we need large cardinals in order to control the value of cardinal characteristics.

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- Every regular uncountable cardinal κ carries uniform ultrafilters (extend the dual filter of $[\kappa]^{<\kappa}$ to an ultrafilter using Zorn's lemma).
- $\kappa^+ \leqslant \mathfrak{u}(\kappa) \leqslant 2^\kappa$.
- It is unclear how to control the base of an arbitrary uniform ultrafilter with forcing.
- That is why we prefer to work with measurable cardinals.

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- For the $(\leftarrow)$ direction, if $j : V \to M$ has critical point κ, then we can define an ultrafilter U on κ by

$$X \in U \iff \kappa \in j(X).$$

- It is easy to see that U is uniform.

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- It is easy to see that U is uniform.
- Since there is a plethora of results about lifting embeddings in forcing extensions, it seems more promising to control the ultrafilter number of a measurable cardinal.

Controlling the ultrafilter number

Theorem (Brooke-Taylor, Fischer, Friedman, Montoya - 2017)

Suppose $\kappa < \kappa^* \leqslant \lambda$ are regular cardinals and κ is supercompact. Then, there is a forcing extension inside which κ remains supercompact, $\mathfrak{u}(\kappa) = \kappa^*$ and $2^\kappa = \lambda$.

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- Supercompactness is a much stronger property than measurability.
- The proof relies on the indestructibility of supercompact cardinals.
- It is worth asking if the large cardinal assumption can be reduced.
- The forcing notion used is an iteration of Mathias forcing.

Iterated Mathias forcing

For a regular $\kappa > \omega$ and U an ultrafilter on κ,

$$\mathbb{M}_U^\kappa = \{\langle s, A\rangle \mid s \in \kappa^{<\kappa}, A \in U\}$$

$$\langle t, B\rangle \leqslant \langle s, A\rangle \text{ iff } t \supseteq s,\ B \subseteq A \text{ and } t - s \in A.$$

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We do an iteration $\mathbb{M}$ where at each stage an ultrafilter U is chosen in order to force with $\mathbb{M}^{\kappa}_{U}$. We allow the generic filter to decide which ultrafilter to use, i.e. we force with the lottery sum

$$\bigoplus_{U \text{ ultrafilter on } \kappa} \mathbb{M}^{\kappa}_{U}$$

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Key Lemma (BT-F-F-M)

Subject to technical assumptions on the iteration, if U is an ultrafilter on κ in $V^{\mathbb{M}}$, then it has been chosen quite often in the lottery.

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Assume the existence of a supercompact cardinal, it is consistent that the first strongly compact cardinal κ is also the first measurable cardinal and indestructible under κ-directed closed forcing.

- Note that under GCH, the first measurable does not possess any degree of supercompactness!
- We aim to adapt the proof of the previous theorem in the Apter-Gitik model.
- Note that this does not improve the consistency strength, but is rather an indication of a possible improvement.

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4. In the final model, κ has no measurable cardinals below it. With Prikry-type arguments we can show it remains strongly compact.
5. With Laver-style arguments we can show that κ becomes indestructible under κ-directed closed forcing. $\square$

Controlling the ultrafilter number

of the first strongly compact cardinal

Theorem (D., 2018)

Suppose κ is a supercompact cardinal and $\kappa < \kappa^* \leqslant \lambda$ are regular cardinals with $\lambda^\kappa = \lambda$ and $\lambda^\kappa = \lambda$. Then, there is a forcing extension in which κ is the first strongly compact and the first measurable, $\mathfrak{u}(\kappa) = \kappa^*$ and $2^\kappa = \lambda$.

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Proof sketch:

- Start with the Apter-Gitik variation of Laver preparation $\mathbb{P}$ that makes κ the first strongly compact and indestructible (iterated Prikry forcing is involved).

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- As κ is indestructible, we can further force with the Mathias iteration $\mathbb{M}$ of BT-F-F-M.
- For a large enough λ, choose a λ-supercompactness embedding $j : V \to M$ with

$$j(f)(\kappa) = \mathbb{M}.$$

- By elementarity,
  $j(\mathbb{P} * \mathbb{M}) \simeq \mathbb{P} * \mathbb{M} * \text{ (Prikry iteration) } * \text{ (tail of } j(\mathbb{P})) * j(\mathbb{M})$.

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- The previous step shows that $\mathfrak{u}(\kappa) \leqslant \kappa^*$. For the converse, we use the fact that the Mathias generics form an unbounded family. $\square$

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Corollary

In the forcing extension constructed before,
$add(\mathcal{M}_\kappa) = cof(\mathcal{M}_\kappa) = non(\mathcal{M}_\kappa) = cov(\mathcal{M}_\kappa) = \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{u}(\kappa)$.

- Not known (to my knowledge) if the cardinals in Cichoń's diagram can be controlled independently to the ultrafilter number.

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- It is still open whether we can violate GCH at a strongly compact, without assuming supercompactness (κ^{++}-supercompactness is the best known consistency bound).
- Some specific cases may be easier to handle (e.g. a measurable limit of supercompact cardinals).

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- We need more than measurability (to violate GCH at a measurable cardinal we need at least a Mitchell rank 2 measurable.)
- Strong cardinals would be a good candidate: we can violate GCH and there is a certain degree of indestructibility.

Thank you for listening!